Analytical Geometry

In exercise 1-3, graph the conic and identify any vertices and foci.

1. $y^2 - 8x = 0$

When written in the standard parabola form, it will be as follows:

$y^2 = 8x$

This equation is in the form $y^2 = 4px$ which indicates that it has a horizontal axis and vertex at (0, 0). The variable p is the distance between the focus and the vertex.

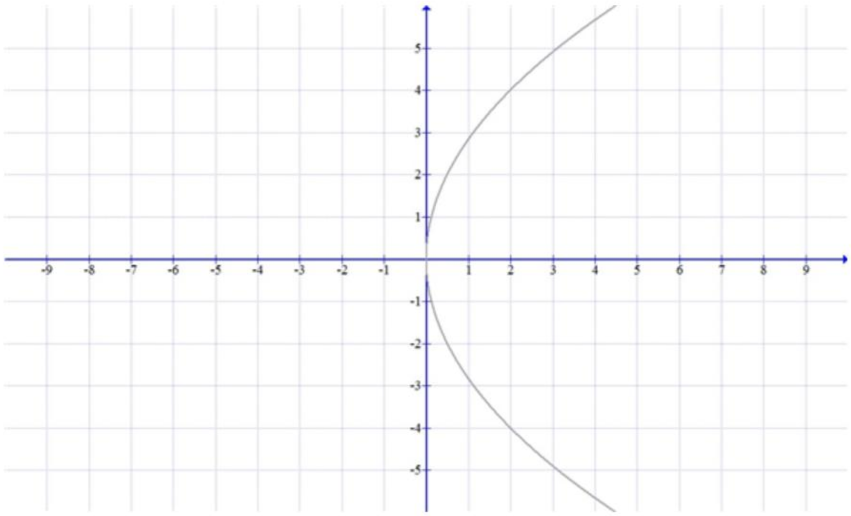
The equations $y^2 = 8x$ and $y^2 = 4px$ can be compared to compute the value of p

$4p = 8$

$p = 2$

Thus the focus of the parabola is (2, 0) and the vertex is (0, 0)

The equation can be graphed as follows:



2. $y^2 - 4x + 4 = 0$

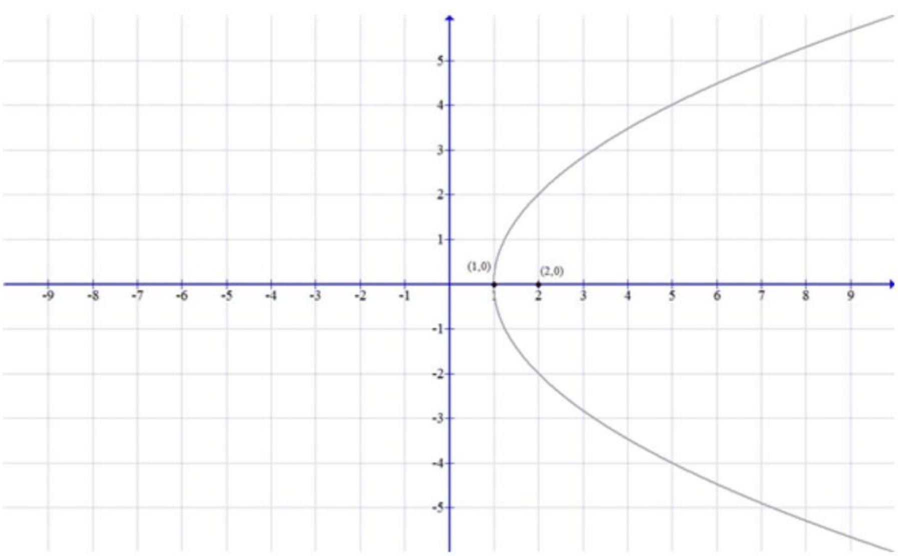
This equation can be rewritten as: $y^2 = 4(x - 1)$

This is in standard parabola form of $(y - \beta) = 4a(x - \alpha)$

vertex is (α, β) which is (1, 0)

and foci is $(\alpha + a, \beta)$, which is (2, 0)

The parabola equation can be graphed as follows:



3. $x^2 - 4y^2 - 4x = 0$

The equation can be rewritten as

$(x^2 - 4x) - 4y^2 = 0$

$x - 2)^2 - 4y^2 = 4$

Both sides can be divided by 4 as follows:

$\frac{(x-2)^2}{4} - \frac{4y^2}{4}$ **4 /4**

$\frac{(x-2)^2}{4} - \frac{(y-0)^2}{1} = 1$

This equation is now in the form

$\frac{(x-h)^2}{a} + \frac{(y-k)^2}{b} = 1$ **which is the standard equation of a hyperbola where the transverse axis is horizontal and (h, k) is the center. The vertices are (h ± a, k)**

The value of h is 2 and k is 0

Therefore, the center (h, k) is (2, 0)

The values of h, k, and a can be substituted in h-a and h + a to find the vertices

The vertices will be (0, 0) and (4, 0)

We can find c using the formula

$c = \pm \sqrt{a^2 + b^2}$

a = 2 and b = 1

c = ± $\sqrt{2^2 + 1^2}$

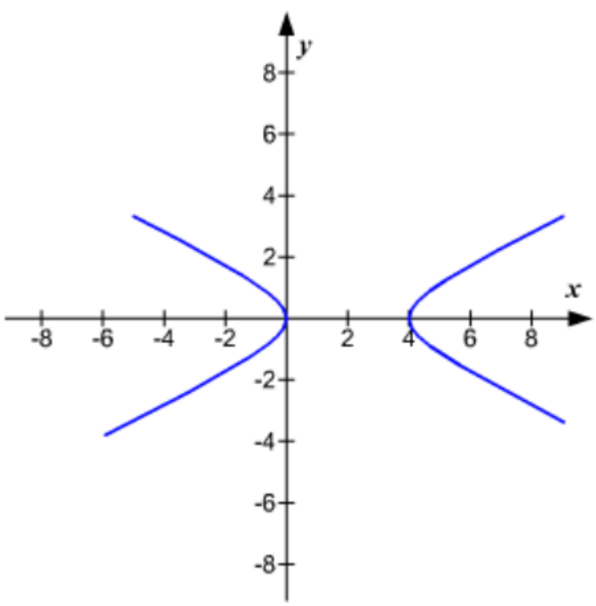
c = ± $\sqrt{5}$

The foci for a hyperbola with a horizontal transverse axis are (h – c, k) and (h + c, k)

(2 - $\sqrt{5}$, 0) and (2 + $\sqrt{5}$, 0)

The foci are (2 - $\sqrt{5}$, 0) and (2 + $\sqrt{5}$, 0)

The hyperbola can be graphed as follows:



4. Find the standard form of the equation of the parabola with focus (8, -2) and directrix x = 4, and sketch the parabola.

parabola focus is (8, -2)

Directrix is x = 4

The axis is perpendicular to the directrix

Axis is a horizontal line and therefore the equation of the parabola can be taken in the form: $(y - k)^2 = 4p\,(x - h)$

where the focus lies on axis at p units from vertex.

From the equation of the parabola, vertex is (h, k), focus is (h + p, k), and directrix is $x = h - p$

From the above,

$h + p = 8$

$k = -2$

$h - p = 4$

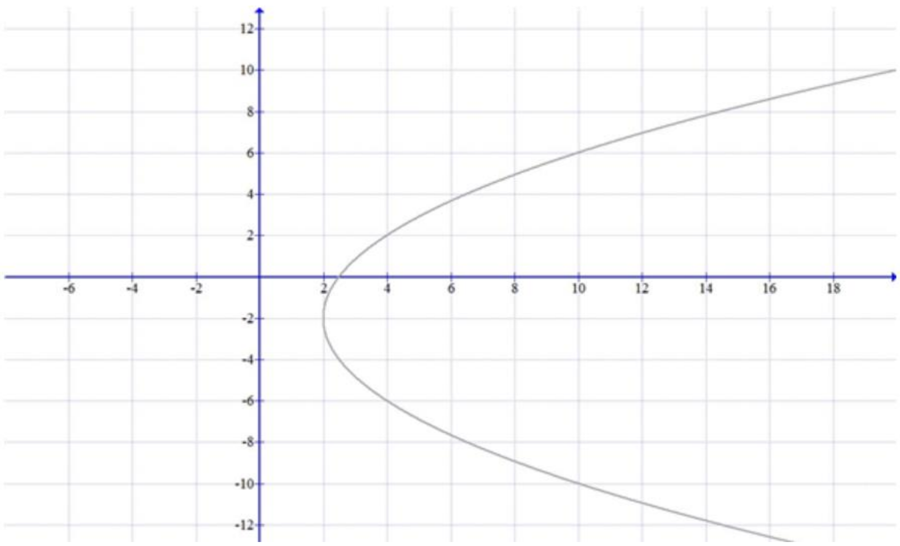
By solving them, we get h = 6, k = 2, and p = 2

These values can then be substituted into the equation of the parabola

$(y + 2)^2 = 4\,(2)\,(x - 6)$

$(y + 2)^2 = 8(x - 6)$

The graph of this parabola equation is as follows:



5. Find the standard form of the equation of the ellipse shown at the right.

Based on the graphed ellipse, the vertices are (-6, 10) and (-6, -4)

The ends of the minor axis are (-10, 3) and (-2, 3)

The x coordinates are equal and therefore the major axis is parallel to the y axis.

The equation of the ellipse can be taken as

$$\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1$$

The distance between two points A (x1, y1) and B (x2, y2) is equal to

AB = $\sqrt{(x2 - x1)^2 + (y2 - y1)^2}$

The length of the major axis is equal to the distance between the vertices and this distance is calculated as follows:

2a = $\sqrt{(-6 + 6)^2 + (10 + 4)^2}$

2a = 14

a = 7

The length of the minor axis is equal to distance between the ends of the minor axis, which is calculated as follows:

2b = $\sqrt{(-10+2)^2+(3-3)^2}$

2b = 8

b = 4

The center of the ellipse is the midpoint of the line segment joining the vertices, which is: $(\frac{-6-6}{2}, \frac{10-4}{2})$ **= (-6, 3)**

Therefore, the center of the ellipse (h, k) is (-6, 3)

The values of h, k, a, b can then be substituted into the equation of the ellipse as follows:

$$\frac{(x+6)^2}{4^2}+\frac{(y-3)^2}{7^2}=1$$

$$\frac{(x+6)^2}{16}+\frac{(y-3)^2}{49}=1$$

6. Find the standard form of the equation of the hyperbola with vertices (0, ±3) and asymptotes $y = \pm\frac{3}{2}x$.

we have $y = \pm\frac{3}{2}x$

so, b/a = 3/2; b = 3a/2

equation of hyperbola: $x^2/a^2 - y^2/b^2 = 1$

$x^2/a^2 - y^2/(3a/2)^2 = 1$ ----> $x^2/a^2 - 4y^2/9a^2 = 1$

vertices: (0, +/-3)

$0/a^2 - 4*9/9a^2 = 1$ ----> $a^2 = 4$

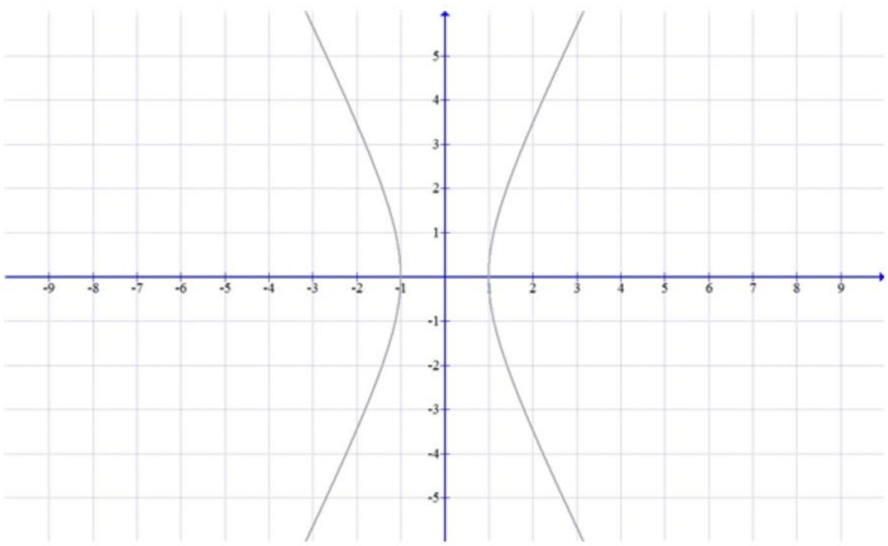
$b^2 = 9a^2/4 = -9*4/4 = -9$

So, $y^2/9 - x^2/4 = 1$

Therefore, the equation of the parabola is $\frac{y^2}{9} - \frac{x^2}{4} = 1$

7. Use a graphing utility to graph the conic $x^2 - \frac{y^2}{4} = 1$. Describe your viewing window.

The conic $x^2 - \frac{y^2}{4} = 1$ represents a hyperbola, which can be graphed as follows:



The above graph is in the window described as:

$-5 \le x \le 5$, $-3 \le y \le 3$

8. a) Determine the number of degrees the axis must be rotated to eliminate the xy- term

of the conic $x^2 + 6xy + y^2 - 6 = 0$

The general second degree equation $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ can be written as

$A'(x')^2 + C'(y')^2 + D'(x') + E'(y') + F' = 0$

We can rotate the coordinate axes through an angle θ, where $\cot 2\alpha = \frac{A-C}{B}$, $0^{o} < 2\alpha < 180^{o}$

The equation $x^2 + 6xy + y^2 - 6 = 0$ can be compared with the second degree equation

$A = 1$, $B = -10$, $C = 1$, $D = 0$, $E = 0$, and $F = -6$

The angle of rotation can be found using the formula

$$\cot 2\alpha = \frac{A-C}{B}$$

$$\cot 2\alpha = \frac{1-1}{-10}$$

$$\cot 2\alpha = 0$$

$$2\alpha = \cot^{-1} 0$$

$$2\alpha = 90$$

$$\alpha = 45^{o}$$

Therefore, the axes are rotated 45^{o} to eliminate the xy term of the conic

b) Graph the conic in part (a) and use a graphing utility to confirm your result.

The equation of the conic in the rotated x'y' coordinate system is

$A' = A\ \mathrm{Cos}^2\alpha + B\cos\alpha\sin\alpha + C\sin^2\alpha$

$C' = A\sin^2\alpha + B\cos\alpha\sin\alpha + C\cos^2\alpha$

$D' = D\cos\alpha + E\sin\alpha$

$E' = -D\sin\alpha + E\cos\alpha$

$F' = F$

$\alpha = 45^o$

Sin 45 = $\frac{1}{\sqrt{2}}$ **and cos 45 =** $\frac{1}{\sqrt{2}}$

We can determine the coefficients of A', C', D', E', F'

$A' = 1.\ (\frac{1}{\sqrt{2}})^2 - 4.\ \frac{1}{\sqrt{2}}\ .\ \frac{1}{\sqrt{2}} + 1.\ (\frac{1}{\sqrt{2}})^2$

$= -1$

$C' = 1.\ (\frac{1}{\sqrt{2}})^2 - (-4).\ \frac{1}{\sqrt{2}}\ .\ \frac{1}{\sqrt{2}} + 1.\ (\frac{1}{\sqrt{2}})^2$

$= 3$

Values of D', E', and F' will be 0, 0, -6 respectively.

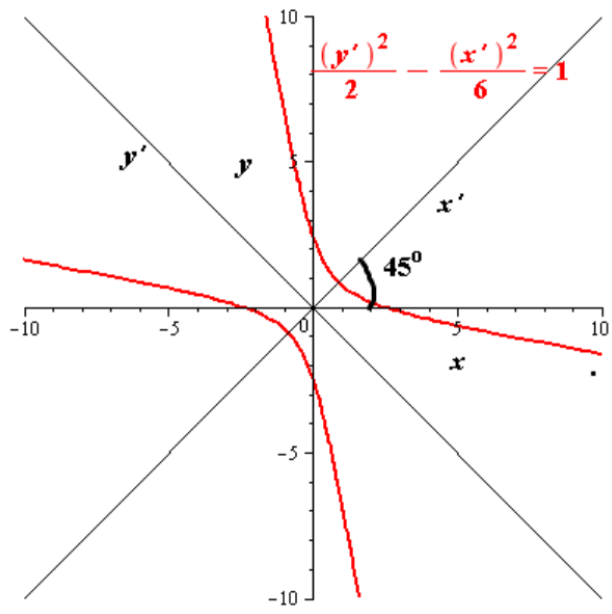
The equation of the conic in the xy plane is

$-1\,(x')^2 + 3\,(y')^2 - 6 = 0$

$-1\,(x')^2 + 3\,(y')^2 = 6$

$\frac{(y')^2}{2} - \frac{(x')^2}{6} = 1$

This is the equation of a hyperbola whose center is the origin of a x'y' coordinate system having a transverse axis on the x axis



9. Solve the system of equations at the right algebraically by the method of substitution. Then verify your results using a graphing utility.

$x^2 + 2y^2 - 4x + 6y - 5 = 0$

$x + y + 5 = 0$

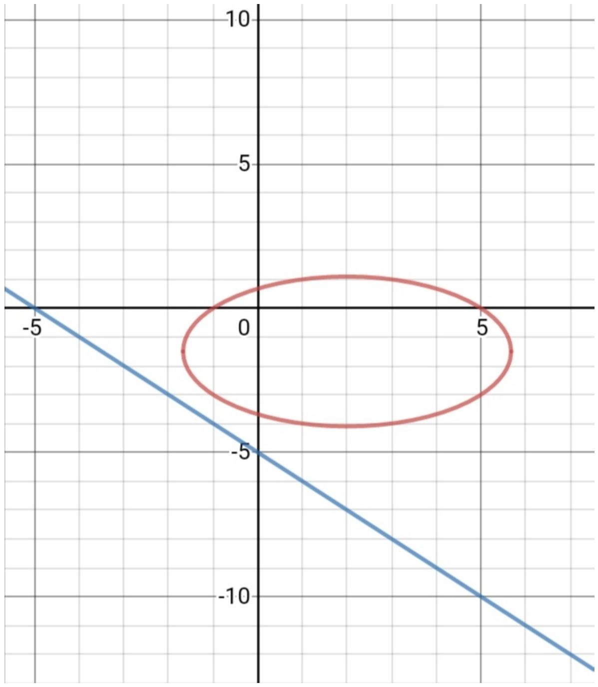
From equation 2, we get $y = -(x + 5)$

$x^2 + 2(-(x + 5))^2 - 4x + 6(-(x + 5)) - 5 = 0$

$x^2 + 2(-x - 5)^2 - 4x - 6(x + 5) - 5 = 0$

$x^2 + 2(x^2 + 25 + 10x) - 4x - 6x - 30 - 5 = 0$

$3x^2 + 10x + 15 = 0$

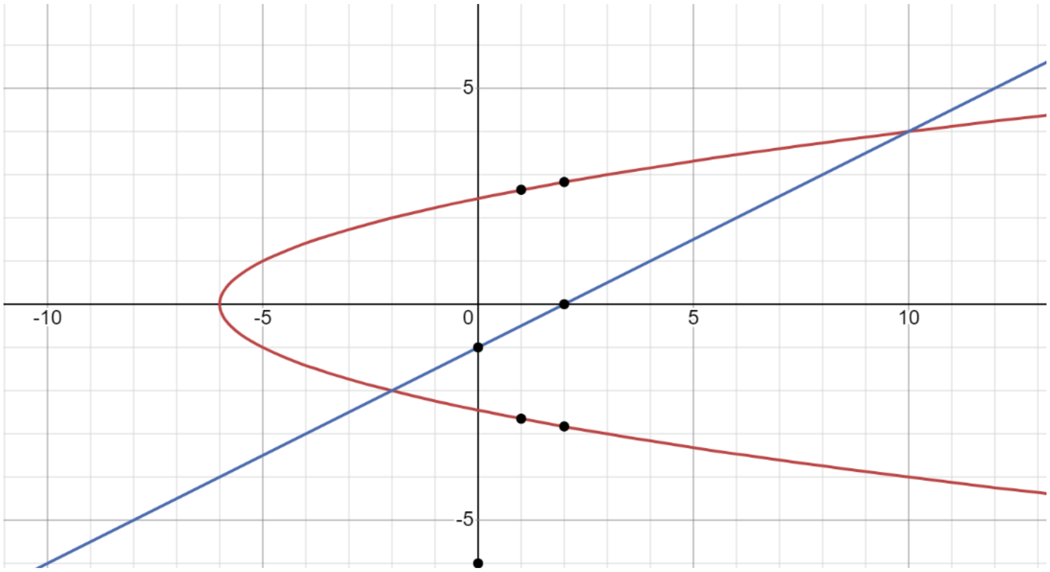


In exercise 10-12, sketch the curve represented by the parametric equations. Then eliminate the parameter and write the corresponding rectangular equation whose graph represent the curve.

10. $x = t^2 - 6$

$y = 1/2t - 1$

The parametric equations can be graphed as follows:



From equation 1:

$x = t^2 - 6$

$x + 6 = t^2$

$\sqrt{x+6} = t$

This value of t can be substituted in equation 2

From equation 2:

$y = 1/2t - 1$

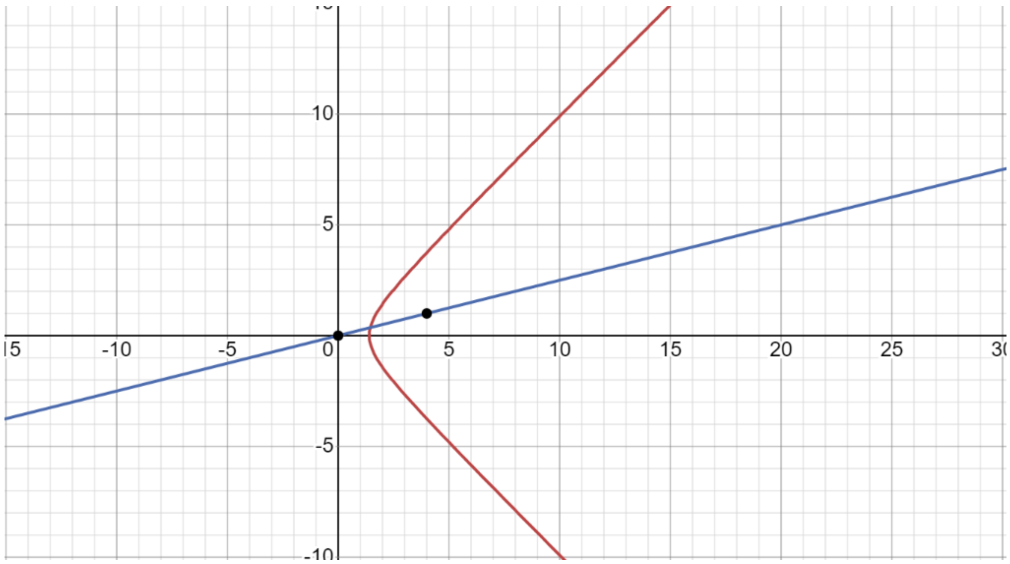
$y = ½ (\sqrt{x+6})$

Therefore, the rectangular equation is $y = \frac{\sqrt{x+6}}{2}$

11. $x = \sqrt{t^2+2}$

$y = t/4$

The parametric equations can be graphed as follows:



From equation 1, we can make t the subject:

x = $\sqrt{t^2 + 2}$

$x^2 = t^2 + 2$

$x^2 - 2 = t^2$

$\sqrt{x^2 - 2}$ = t

We then substitute the value of t in equation 2:

y = t/4

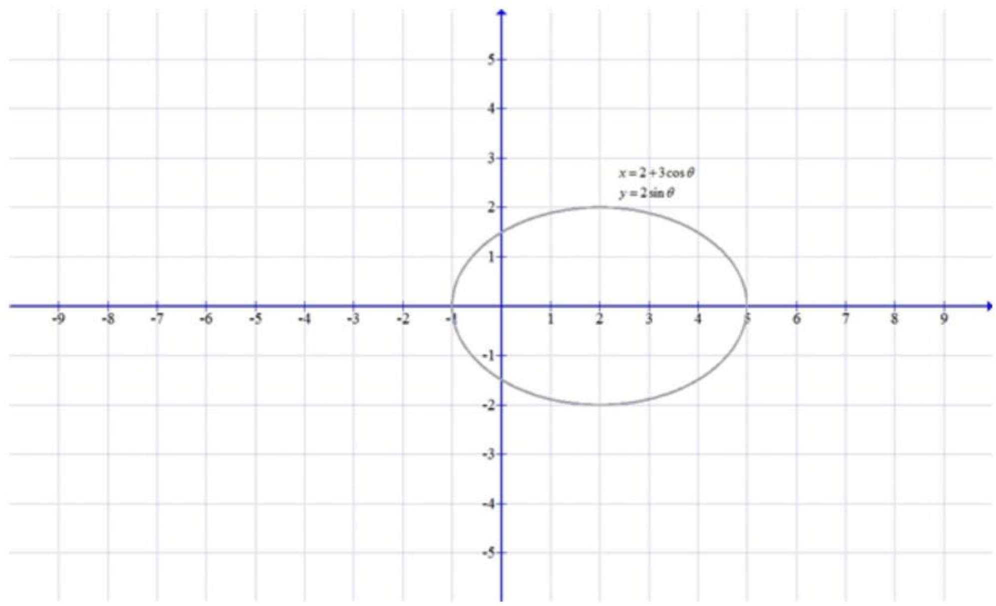
y = $\frac{\sqrt{x^2-2}}{4}$

Therefore, the rectangular equation is y = $\frac{\sqrt{x^2-2}}{4}$

12. x = 2 + 3 cos θ

$y = 2 \sin \theta$

The parametric equations can be graphed as follows:



We can find the rectangular equation to set of parametric equations as follows:

$\frac{x-2}{3} = \cos \theta$

$\frac{y}{2} = \sin \theta$

$\frac{(x-2)^2}{9} + \frac{y^2}{4} = 1$

Hence, the corresponding parametric equation is $\frac{(x-2)^2}{9} + \frac{y^2}{4} = 1$

In exercises 13-15, find two different sets of parametric equations for the given rectangular equation. (There are many correct equations).

13. $y = x^2 + 10$

First set of parametric equations can be:

$x = t$

$y = t^2 + 10$

Second set could be

$x = \frac{t}{2}$

$y = \frac{t^2}{4} + 10$

14. $x + y^2 = 4$

First set of parametric equations could be:

$x = t$

$y = \sqrt{4 - t}$

Second set of parametric equations is:

$y = t$

$x = 4 - t^2$

15. $x^2 + 4y^2 - 16 = 0$

First set of parametric equations is:

$x = t$

$y = \frac{\sqrt{16 - x^2}}{4}$

Second set of parametric equations is:

x = 4t

y = 2 $\sqrt{1-t^2}$

16. Convert the polar coordinate (-14, 5π/3) to rectangular form.

θ = $\frac{5\pi}{3}$

θ can be written as tan⁻¹ $(\frac{y}{x})$

$\tan^{-1}(\frac{y}{x}) = \frac{5\pi}{3}$

$\frac{y}{x} = \tan(\frac{5\pi}{3})$

y = - $\sqrt{3x}$

17. Convert the rectangular coordinate (2, -2) to polar form and find two additional polar representations of this point. (There are many correct answers)

The rectangular coordinates (x, y) and the polar coordinates (r, **θ) are related.**

To convert the coordinates from rectangular to polar, the following steps are taken:

r = $\sqrt{x^2+y^2}$

α = arc tan $(\left|\frac{x}{y}\right|)$

if both x and y are positive, then θ = α

if x is negative and y is positive then θ = π – α

if both x and y are negative, then $\theta = \pi + \alpha$

If x is positive and y is negative, then $\theta = -\alpha$

By using the above conversions, the corresponding rectangular coordinates of the given point can be computed as follows:

$r = \sqrt{2^2 + (2)^2}$

$r = \sqrt{8}$

$r = 2\sqrt{2}$

The directed angle of the point is given as follows, while noting that x is positive and y negative

$\theta = \arctan\left(\left|\frac{-2}{2}\right|\right)$

$= \arctan(1)$

$= -\frac{\pi}{4}$

Therefore, the polar coordinates of (2, -2) is $\left(2\sqrt{2}, -\frac{\pi}{4}\right)$

We can find another set of polar coordinates for $0 \leq \theta < 2\pi$

point (r, θ) **can be represented as:**

$(r, \theta) = (r, \theta \pm \pi)$

Therefore, another set of coordinates to the point ($2\sqrt{2}$, $-\frac{\pi}{4}$) is ($-2\sqrt{2}$, $-\frac{\pi}{4} + \pi$) or ($-2\sqrt{2}$, $\frac{3\pi}{4}$)

Therefore, the two sets of polar coordinates of the point whose rectangular coordinates are (2, -2) are ($2\sqrt{2}$, $-\frac{\pi}{4}$) and ($-2\sqrt{2}$, $\frac{3\pi}{4}$)

18. Convert the rectangular equation $x^2 + y^2 - 12y = 0$ to polar form.

The relationship between polar and rectangular coordinates are as follows:

$x = r \cos \theta$

$y = r \sin \theta$

$r^2 = x^2 + y^2$

In the given equation substitute the x and y with their respective values

$r^2 = 12 (r \sin \theta)$

$r^2 = 12 r \sin \theta$

$r = 12 \sin \theta$

19. Convert the polar equation $r = 2 \sin \theta$ to rectangular form.

The relationship between polar and rectangular coordinates are as follows:

$x = r \cos \theta$

$y = r \sin \theta$

$\mathbf{r^2 = x^2 + y^2}$

$\boldsymbol{\theta} = \mathbf{tan^{-1}}\left(\frac{y}{x}\right)$

$\mathbf{r^2 = 2r \sin \theta}$

$\mathbf{x^2 + y^2 = 2y}$

$\mathbf{x^2 + y^2 - 2y = 0}$

$\mathbf{x^2 + (y - 1)^2 = 1}$

In exercises 20-22, identify the conic represented by the polar equation algebraically.

Then use a graphing utility to graph the polar equation.

20. $r = 2 + 3 \sin \theta$

The graphs of polar equation

$\mathbf{r = a \pm b \cos \theta}$

$\mathbf{r = a \pm b \sin \theta}$ **are Lima cons**

A polar equation represents a Limacon with inner loop when

$\mathbf{a/b < 1}$

It represents Cardioid when

$\mathbf{a/b = 1}$

It represents Dimpled Limacon when

$\mathbf{1 < a/b < 2}$

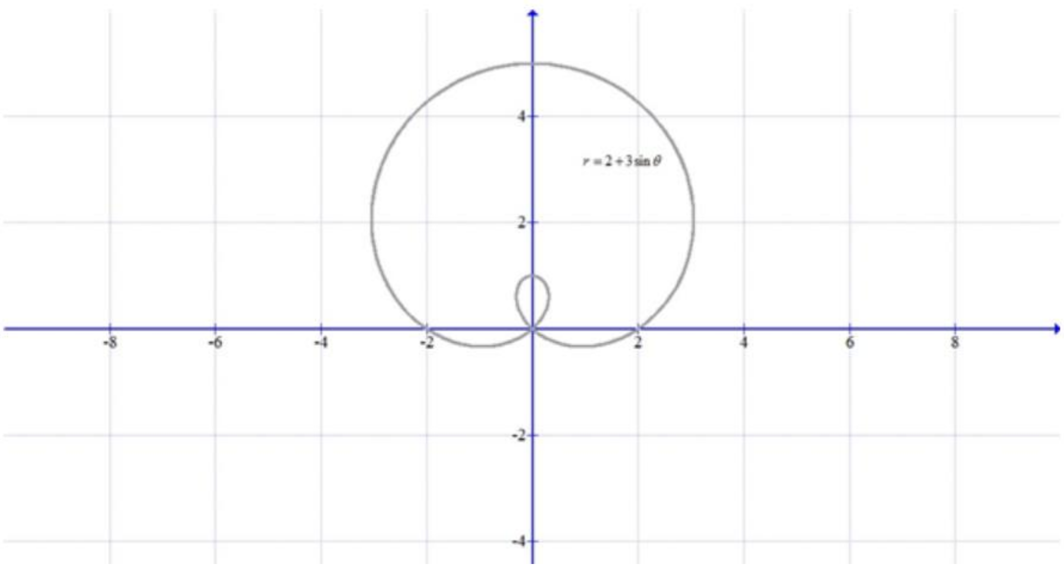
It represents convex Limacon when a/b = 2

In the given equation, r = 2 + 3 sin θ

a = 2 and b = 3

a/b = 2/3 < 1

Therefore, the polar equation r = 2 + 3 sin θ represents a Limacon with inner loop.



21. $r = \frac{1}{1-\cos\theta}$

r (1 – cos θ) = 1

r – r cos θ = 1

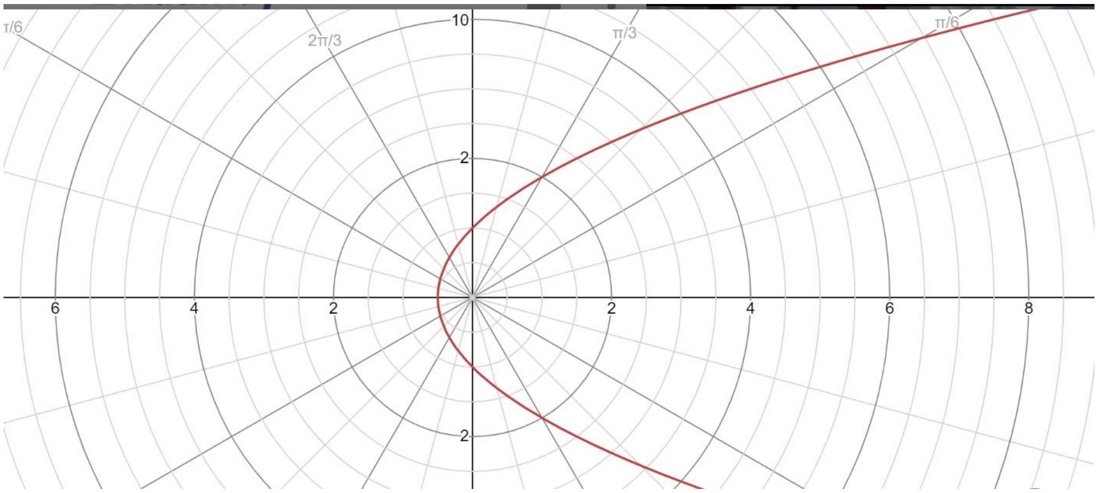
$\sqrt{x^2+y^2}$ **– x = 1**

$x^2 + y^2 = (1 + x)^2$

$x^2 + y^2 = 1 + x^2 + 2x$

$y^2 = 1 + 2x$

This is a parabola and not a conic



22. $r = \frac{4}{2 + 3 \sin \theta}$

The relationship between polar and rectangular coordinates are as follows:

$x = r \cos \theta$

$y = r \sin \theta$

$r^2 = x^2 + y^2$

$\theta = \tan^{-1}\left(\frac{y}{x}\right)$

$r\ (2 + 3 \sin \boldsymbol{\theta}) = \mathbf{4}$

$2r + 3r \sin \theta = 4$

$2r + 3\ (y) = 4$

$2\left(\sqrt{x^2 + y^2}\right) + 3y = 4$

$$2\left(\sqrt{x^2 + y^2}\right) = 4 - 3y$$

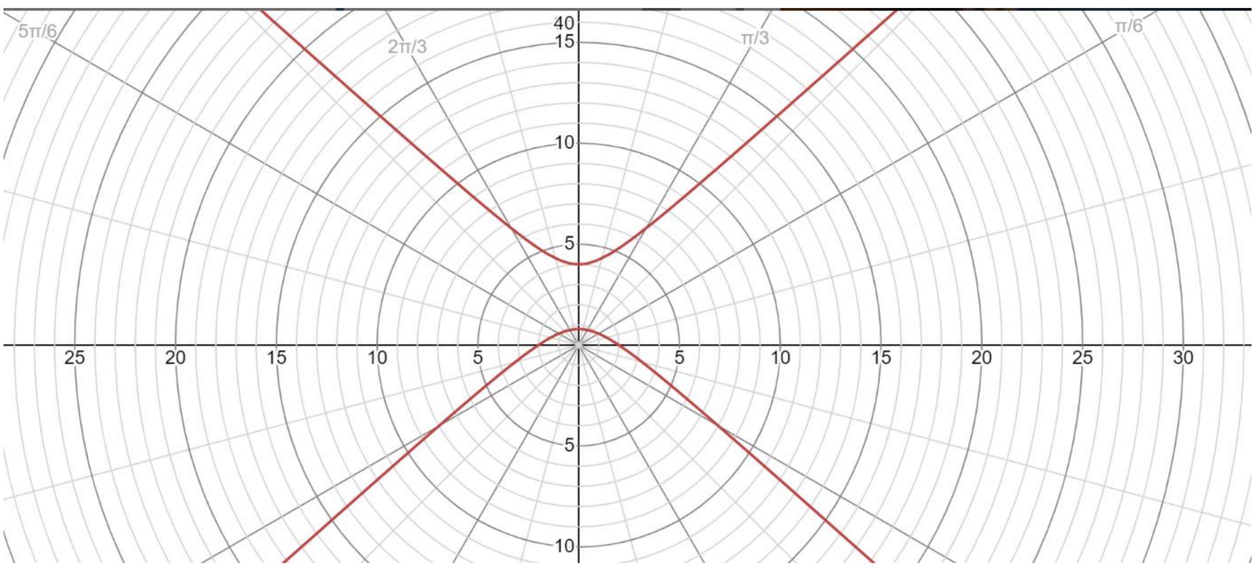
$$2(x^2 + y^2) = (4 - 3y)^2$$

$$2(x^2 + y^2) = 16 - 24y + 9y^2$$

$$2x^2 + 2y^2 = 16 - 24y + 9y^2$$

$$2x^2 = 16 - 24y + 7y^2$$

It is not a conic



23. Find a polar equation of an ellipse with its focus at the pole, an eccentricity of e = ¼, and directrix at y = 4

Since the directrix is horizontal and above the pole, the general equation of the conic will be as follows:

$$r = \frac{ep}{1+e\sin\theta}$$

The distance from the focus (pole) to the directrix is p = 4

These values of e and p can be substituted in the equation as follows:

$$\mathbf{r} = \frac{\frac{1}{4}(4)}{1+(\frac{1}{4})\sin\theta}$$

$$\mathbf{r} = \frac{4}{4+\sin\theta}$$

Therefore, the equation of the required ellipse is r = $\frac{4}{4+\sin\theta}$

24. Find a polar equation of a hyperbola with its focus at the pole, an eccentricity of e = 5/4, and a directrix at y = 2

Since the directrix is horizontal and above the pole, the general equation of the conic will be as follows:

$$\mathbf{r} = \frac{ep}{1+e\sin\theta}$$

The distance from the focus (pole) to the directrix is p = 2

These values of e and p can be substituted in the equation as follows:

$$\mathbf{r} = \frac{\frac{5}{4}(2)}{1+(\frac{5}{4})\sin\theta}$$

$$\mathbf{r} = \frac{10}{4+5\sin\theta}$$

Therefore, the equation of the required hyperbola is r = $\frac{10}{4+5\sin\theta}$

25. For polar equation r = 8 cos 3θ, find the maximum value of $|r|$ and any zeros of r. Verify your answers numerically.

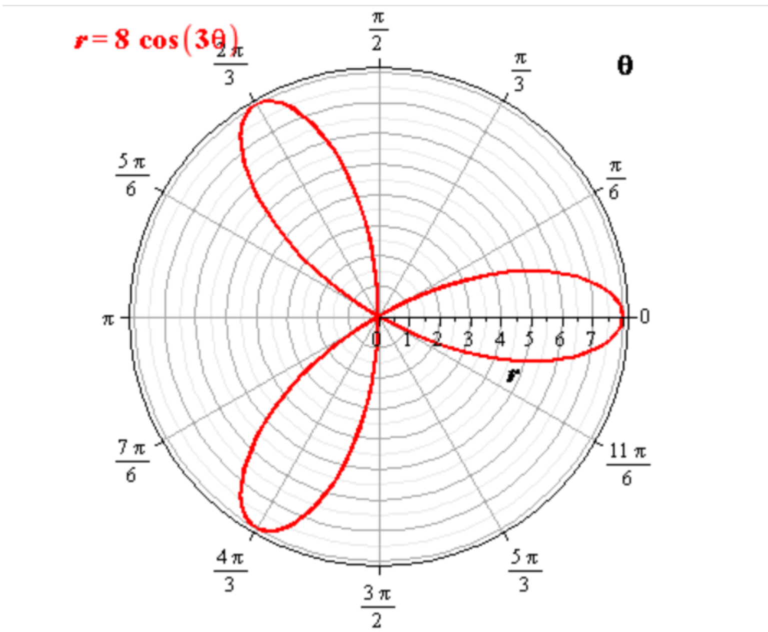
The curve is of the form r = a cos nθ

where n = 3

Therefore, it is a rose curve with 3 leaves.

r = 8 cos 3 (-θ) and r = 8 cos 3θ meaning the curve is symmetrical with respect to the polar axis.

The graph sketch is as shown below:



From the sketch, the maximum absolute value of r is 8 while the zeros of r include

$\frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}$

The maximum value for cos 3θ = 1

So the maximum value of r is 8 (1) = 8

For zeros of r,

r = 0

8 cos 3θ = 0

cos 3θ = 0

3θ = $\frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}$

Multiply by 1/3 to each term to give:

$\frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}$

So the zeros of r are $\frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}$